

MSC2010: 13D05, 14A05

A NOTE ON HOMOLOGICAL DIMENSION OF A FAMILY OF COHERENT SHEAVES

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We prove a theorem on how a conclusion on homological dimension of the family of coherent sheaves on a scheme can be done from homological dimension of the restriction of this family to the reduction of the base.

Bibliography: 5 items.

Keywords: algebraic coherent sheaves, homological dimension, flat module.

To the blessed memory of my Mum

The problem we solve in the present note is

how to conclude about homological dimension of the family of sheaves on the family of schemes if it is known about homological dimension of the reduction?

As usually if T is a scheme with structure sheaf \mathcal{O}_T then its *reduction* is a scheme consisting of the same topological space but with structure sheaf equal to $\mathcal{O}_{T_{\text{red}}} := \mathcal{O}_T / \text{Nil}(\mathcal{O}_T)$ where $\text{Nil}(\mathcal{O}_T)$ is nilradical of \mathcal{O}_T . This is called for brevity a *reduced scheme structure*. The \mathcal{O}_T -module epimorphism onto the quotient module sheaf $\mathcal{O}_T \twoheadrightarrow \mathcal{O}_{T_{\text{red}}}$ induces a canonical closed immersion $T_{\text{red}} \hookrightarrow T$ of schemes. From now T is a base scheme of a flat morphism of finite type $f : X \rightarrow T$ of Noetherian schemes. We introduce notations $X_{\text{red}} := X \times_T T_{\text{red}}$, $f_{\text{red}} : X_{\text{red}} \rightarrow T_{\text{red}}$, i.e. X_{red} is a restriction of the family X to the reduction T_{red} as to a closed subscheme in T . Actually the scheme X_{red} can be nonreduced but f_{red} is flat morphism as a morphism obtained by a base change of a flat morphism. Now let \mathbb{E} be a coherent \mathcal{O}_X -module and let \mathbb{E} be flat as \mathcal{O}_T -module. This is a standard situation in various problems of algebraic geometry when families of sheaves are considered, especially in moduli problems. Denote $\mathbb{E}_{\text{red}} = \mathbb{E} \otimes_{\mathcal{O}_T} \mathcal{O}_{T_{\text{red}}}$.

Let A be a commutative ring, M A -module. Also introduce parallel algebraic notations: a *reduction* $A_{\text{red}} := A / \text{Nil} A$ of the ring A is its quotient ring over its nilradical. If a commutative ring B is an A -algebra then we denote $B \otimes_A A_{\text{red}} =: B_{\text{red}}$. Hence B_{red} is not obliged to be reduced but it is A_{red} -flat whenever B is A -flat, by well-known change-of-ring theorem. Also denote $M \otimes_A A_{\text{red}} =: M_{\text{red}}$.

- (1) for all A -modules N $\text{Ext}_A^{n+1}(M, N) = 0$,
- (2) in any exact A -sequence $0 \rightarrow E_n \rightarrow E_{n-1} \rightarrow \cdots \rightarrow E_0 \rightarrow M \rightarrow 0$ if E_j are projective for $0 \leq j \leq n-1$ then E_n is also projective,
- (3) there is a projective A -resolution of length n , i.e. there exists an exact sequence

$$0 \rightarrow E_n \rightarrow \cdots \rightarrow E_1 \rightarrow E_0 \rightarrow M \rightarrow 0$$

with E_j projective for $0 \leq j \leq n$.

Since if A is local ring then any projective A -module is free [1, theorem 19.2 and comment thereafter]. Then when working with coherent sheaves on schemes we speak of locally free resolutions instead of projective ones.

As usually the symbol $\mathrm{hd}_X \mathbb{E}$ means homological dimension of the coherent sheaf \mathbb{E} as \mathcal{O}_X -module.

We prove the following well-expected result.

Theorem 1. Algebraic version. *Let $f^\sharp : A \rightarrow B$ be local homomorphism of local Noetherian rings and B is flat as A -algebra. Let M is B -module of finite type which is flat over A . Then following assertions are equivalent:*

- 1) $\text{hd}_B M \leq n$,
- 2) $\text{hd}_{B_{\text{red}}} M_{\text{red}} \leq n$.

Theorem 2. Sheaf version. *Let $f : X \rightarrow T$ be a flat morphism of Noetherian schemes, \mathbb{E} coherent \mathcal{O}_X -module which is flat over T . Then following assertions are equivalent:*

- 1) $\text{hd}_X \mathbb{E} \leq n$,
- 2) $\text{hd}_{X_{\text{red}}} \mathbb{E}_{\text{red}} \leq n$.

Remark 1. The case $n = 1$ for a trivial family of schemes over a field was considered in the author's paper [5].

Remark 2. Since both theorems are just versions of the same result their proofs are transferred literally to each other and we prove an algebraic version.

Proof of Theorem 1. For the implication 1) \Rightarrow 2) we do not need locality and consider B -exact sequence

$$0 \rightarrow E_n \rightarrow E_{n-1} \rightarrow \cdots \rightarrow E_1 \rightarrow E_0 \rightarrow M \rightarrow 0$$

with E_j free for $j \geq 0$. Cutting it into triples we have

$$\begin{array}{ccccccc} 0 \rightarrow E_n & \rightarrow & E_{n-1} & \rightarrow & M^{(n-1)} & \rightarrow & 0, \\ \cdot & & \cdot & & \cdot & & \cdot, \\ 0 \rightarrow M^{(j+1)} & \rightarrow & E_j & \rightarrow & M^{(j)} & \rightarrow & 0, \\ \cdot & & \cdot & & \cdot & & \cdot, \\ 0 \rightarrow M^{(1)} & \rightarrow & E_0 & \rightarrow & M & \rightarrow & 0. \end{array}$$

Since M and E_j , $j \geq 0$, are flat A -modules [4, Ch. 3, sect. 7, transitivity (1)] we have that $M^{(j)}$ are A -flat for $j \geq 1$. Hence tensoring by $\otimes_B B_{\text{red}} = \otimes_A A_{\text{red}}$ ("cancelation formula") we have $\text{Tor}_i^A(M^{(j)}, A_{\text{red}}) = \text{Tor}_i^A(M, A_{\text{red}}) = 0$ and come to exact sequence

$$0 \rightarrow E_{n\text{red}} \rightarrow E_{(n-1)\text{red}} \rightarrow \cdots \rightarrow E_{1\text{red}} \rightarrow E_{0\text{red}} \rightarrow M_{\text{red}} \rightarrow 0$$

with $E_{j\text{red}}$ free as B_{red} -modules for $0 \leq j \leq n$.

For the opposite implication we organize induction on n . Assume that the theorem is true for coherent B -sheaves of homological dimension not greater than $n - 1$. Let we are given an B -module epimorphism $E_0 \twoheadrightarrow M$ where E_0 is free and let $M' := \ker(E_0 \twoheadrightarrow M)$. Let $\text{hd}_{B_{\text{red}}} M_{\text{red}} \leq n$. We are to conclude that $\text{hd}_B M \leq n$. By the exact B -triple

$$0 \rightarrow M' \rightarrow E_0 \rightarrow M \rightarrow 0$$

and tensoring by $\otimes_B B_{\text{red}} = \otimes_A A_{\text{red}}$, since M is A -flat then $\text{Tor}_i^A(M, A_{\text{red}}) = 0$ for $i > 0$ and hence we come to the exact A_{red} -triple

$$0 \rightarrow M'_{\text{red}} \rightarrow E_{0\text{red}} \rightarrow M_{\text{red}} \rightarrow 0$$

where $\text{hd}_{A_{\text{red}}} M'_{\text{red}} \leq n - 1$. By flatness of E_0 as B -module and of the homomorphism f^\sharp the term E_0 is A -flat [4, Ch. 3, sect. 7, transitivity (1)] and hence M' is also flat as A -module. By the inductive assumption $\text{hd}_B M' \leq n - 1$ and hence $\text{hd}_B M \leq n$.

For the base of induction set $n = 0$. This means that M is flat as A -module and M_{red} is free as A_{red} -module.

Apply the following result from A. Grothendieck's SGA:

Proposition 1. [2, Ch. IV, Corollaire 5.9] *Let $A \rightarrow B \rightarrow C$ be local homomorphisms of local Noetherian rings, M be C -module of finite type. Assume that B is flat over A and k is a residue field of A . Then following assertions are equivalent:*

- (i) M is B -flat;
- (ii) M is A -flat and $M \otimes_A k$ is $B \otimes_A k$ -flat.

For our purposes set $B \rightarrow C$ to be an identity isomorphism, M is flat over A and M_{red} is free (i.e. flat) as A_{red} -module. Then $M \otimes_A k = M_{\text{red}} \otimes_{A_{\text{red}}} k$, $B \otimes_A k = B_{\text{red}} \otimes_{A_{\text{red}}} k$, and $M_{\text{red}} \otimes_{A_{\text{red}}} k$ is flat over $B_{\text{red}} \otimes_{A_{\text{red}}} k$ because M_{red} is free over B_{red} . From this we conclude that M is free as B -module. This completes the proof of the theorem 1. \square

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